

A hyperkähler structure on the cotangent bundle of a complex Lie group

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1. Introduction

The cotangent bundle T^*G of a compact Lie group G has the structure of a complex manifold by virtue of the fact that it may be identified with G^c , the complex Lie group. To write down such an identification, first identify the cotangent bundle with the tangent bundle using a bi-invariant metric, then trivialize the tangent bundle in a left-invariant manner, so as to have

$$T^*G = G \times \mathfrak{g}$$

where \mathfrak{g} is the Lie algebra. The required isomorphism then results from the map which expresses the polar decomposition of a complex group element:

$$\begin{aligned} G \times \mathfrak{g} &\rightarrow G^c \\ (u, \xi) &\mapsto u \exp(i\xi). \end{aligned}$$

In addition to this complex structure, the cotangent bundle carries, of course, a symplectic form; and the two are compatible, in that they make T^*G a Kähler manifold.

Our purpose here is to explore, briefly, what might be thought of as an analogue of the facts above, with the quaternions replacing the complexes: just as the cotangent bundle of a compact (real) group is complex, so the cotangent bundle of the corresponding complex group is “quaternionic”. More precisely, we exhibit a *hyperkähler* structure on T^*G^c .

Recall that a hyperkähler structure [1, 8] on a manifold M consists of three complex structures I, J, K satisfying the quaternion relations, together with a Riemannian metric h which is Kähler with respect to all

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three. The corresponding Kähler forms $\omega_1, \omega_2, \omega_3$ give M three symplectic structures. If we focus on one complex structure, say I , and combine the other two Kähler forms into the complex form

$$\omega_c = \omega_2 + i\omega_3,$$

then, as it turns out, ω_c is a holomorphic $(2,0)$ -form, of maximal rank, on the complex manifold (M, I) . Thus the problem of discovering a hyperkähler structure can be seen as the problem of finding a very special Kähler metric on a holomorphic-symplectic manifold. Our main result asserts the existence of such a metric in the case that the underlying manifold is T^*G^c :

Proposition 1. *If G is a compact Lie group, the cotangent bundle T^*G^c of the corresponding complex group carries a hyperkähler structure which is invariant under left- and right-translations by elements of G .*

The proof of this proposition which we give in the next section is rather indirect and does not lend itself to any calculation of the metric in general. In the case $G = S^1$, the cotangent bundle T^*G^c is just $S^1 \times \mathbb{R}^3$, and the hyperkähler metric of the proposition is the flat one. In the case $G = \mathrm{SU}(2)$ a direct calculation of the metric based on the description of §2 might just be practicable, but for any more complicated case it seems quite unfeasible. There remains, however, the possibility that the hyperkähler structure admits a simpler and more direct description than the one we have found.

In the third section we shall examine the twistor space of the hyperkähler structure (in the sense of [8], for example). Unlike the metric, this turns out to have a very simple form.

2. Nahm's equations

We consider the ordinary differential equations introduced by Nahm in connection with the classification of “magnetic monopoles” [2, 5, 3]. These are three equations for four Lie-algebra-valued functions of one variable,

$$A_i : \mathbb{R} \rightarrow \mathfrak{g} \quad (i = 0, \dots, 3),$$

and take the form

$$\begin{aligned} \frac{dA_1}{ds} + [A_0, A_1] + [A_2, A_3] &= 0 \\ \frac{dA_2}{ds} + [A_0, A_3] + [A_3, A_1] &= 0 \\ \frac{dA_3}{ds} + [A_0, A_4] + [A_1, A_2] &= 0. \end{aligned} \tag{1}$$

For our case, we take \mathfrak{g} to be the Lie algebra of the compact group G of Proposition 1, and we seek solutions which are smooth on the *closed* interval $[0, 1]$. (This is a little different from the situation relevant to monopoles.) Thus we define Ω to be the Banach space of all C^1 maps from $[0, 1]$ to \mathfrak{g} and we define N to be the solution set of the equations:

$$N = \{ (A_0, \dots, A_3) \in \Omega^4 \mid \text{equations (1) hold} \}.$$

Define the “gauge group” \mathcal{G} as the space of all C^2 maps $g : [0, 1] \rightarrow G$, and let \mathcal{G} act on Ω^4 by

$$\begin{aligned} A_0 &\mapsto gA_0g^{-1} + g\frac{dg^{-1}}{ds} \\ A_i &\mapsto gA_i g^{-1} \quad (i = 1, 2, 3). \end{aligned}$$

(Since G is arbitrary and is not given a matrix representation, it might be better to write $\text{Ad}(g)(A_i)$ in place of $gA_i g^{-1}$; but we will stick with the latter notation.) The equations (1) are invariant under this action, so \mathcal{G} acts also on the solution set N . Take \mathcal{G}_0 to be the normal subgroup of \mathcal{G} consisting of those g for which $g(0) = g(1) = 1$, and define M to be the quotient space

$$M = N/\mathcal{G}_0.$$

The proof of Proposition 1 now involves two steps. We shall show:

- (i) *M is, in a natural way, a hyperkähler manifold;*
- (ii) *with respect to any one of the complex structures, the underlying holomorphic symplectic manifold is T^*G^c .*

Proof of (i). Nahm’s equations arise from the self-dual Yang-Mills equations (on flat space) by dimensional reduction. It is a common feature of the equations that arise in this way (these being Nahm’s equations, the Bogomolny equation and the two-dimensional self-duality equations studied by Hitchin [7]) that the moduli spaces of their solutions have hyperkähler structures. This phenomenon is discussed in [6], where it is explained that, formally at least, it is a consequence of the fact that these moduli spaces are examples of hyperkähler quotients, in the sense of [8]. We run through some of the argument as it applies to our case.

First of all, the Banach space Ω^4 has a (flat) hyperkähler structure if we regard it as $\mathbb{H} \otimes \Omega$, where \mathbb{H} denotes the quaternions. The three complex

structures can be written

$$\begin{aligned} I(A_0, A_1, A_2, A_3) &= (-A_1, A_0, -A_3, A_2) \\ J(A_0, A_1, A_2, A_3) &= (-A_2, A_3, A_0, -A_1) \\ K(A_0, A_1, A_2, A_3) &= (-A_3, -A_2, A_1, A_0) \end{aligned}$$

and the metric h comes from the L^2 norm

$$\|A\|^2 = \int_0^1 \left(\sum_0^3 \langle A_i(s), A_i(s) \rangle \right) ds.$$

At this point we have introduced an invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} .

The left-hand sides of the equations (1) define a smooth map

$$\mu : \Omega^4 \rightarrow \Gamma^3$$

where Γ is the space of C^0 maps $\gamma : [0, 1] \rightarrow \mathfrak{g}$. We want now to show that the solution set $N = \mu^{-1}(0)$ is a smooth Banach submanifold of Ω^4 . By the inverse function theorem, it is enough to show that the derivative $d\mu$ has a right-inverse at each point $A = (A_0, \dots, A_3) \in N$, which means solving for $a = (a_0, \dots, a_3)$ the linear equations

$$\begin{aligned} \frac{da_1}{ds} - [A, Ia] &= \gamma_1 \\ \frac{da_2}{ds} - [A, Ja] &= \gamma_2 \\ \frac{da_3}{ds} - [A, Ka] &= \gamma_3. \end{aligned}$$

(In these equations, $[A, b]$ stands for $\sum_0^3 [A_i, b_i]$.) This presents no problem: the equations have a unique solution satisfying the additional constraints $a_0 \equiv 0$ and $a_i(0) = 0$ ($i = 1, 2, 3$); so the required right-inverse to $d\mu$ exists.

To show that N/\mathcal{G}_0 is a smooth manifold too, we must show that there is a “slice” for the action of \mathcal{G}_0 . Again, let $A \in N$. With respect to the L^2 inner product, the orthogonal complement of the tangent space to the \mathcal{G}_0 -orbit through A is the set of all $A + a$ with a satisfying

$$\frac{da_0}{ds} + [A, a] = 0. \tag{2}$$

We will show that, in a neighborhood of A , every \mathcal{G}_0 -orbit meets this slice once.

So consider the orbit of $A + b$, for some small b . For $g \in \mathcal{G}_0$ we can write

$$g(A + b) = A + a$$

with

$$\begin{aligned} a_0 &= gA_0g^{-1} - A_0 + gb_0g^{-1} + g\frac{dg^{-1}}{ds} \\ a_i &= gA_ig^{-1} - A_i + gb_ig^{-1} \quad (i = 1, 2, 3). \end{aligned}$$

If we put $g = \exp(u)$ and linearize the equation (2), we obtain an equation for u which can be schematically written

$$D^*(Du + [b, u]) = D^*b$$

where $D^* : \Omega^4 \rightarrow \Gamma$ is the operator

$$D^*c = \frac{dc_0}{ds} + [A, c].$$

The operator D^*D is invertible if we impose the boundary conditions $u(0) = u(1) = 0$; so by the inverse function theorem, if b is small, there exists a unique $g \in \mathcal{G}_0$ close to 1 such that $g(A + b)$ lies in the slice defined by (2).

This, in outline, is the proof that M is a smooth manifold. The argument shows also that if $[A] \in M$ represents the orbit of $A \in N$, then the tangent space $T_{[A]}M$ is isomorphic to the solution set of the equations

$$\begin{aligned} \frac{da_0}{ds} + [A, a] &= 0 \\ \frac{da_1}{ds} - [A, Ia] &= 0 \\ \frac{da_2}{ds} - [A, Ja] &= 0 \\ \frac{da_3}{ds} - [A, Ka] &= 0. \end{aligned}$$

The linear map $\Omega^4 \rightarrow \Gamma^4$ defined by the left hand sides of these four equations commutes with the actions of I , J and K . So $T_{[A]}M$ is invariant under I , J , K and this gives M three almost complex structures. By the same route, M acquires a Riemannian metric h from the L^2 metric on Ω^4 .

We shall not give the proof that the complex structures on M are integrable and Kähler. A suitable model for the argument can be found in [7], on page 91. The reason these things work out is that, as we mentioned above, M is a hyperkähler quotient (of Ω^4 by \mathcal{G}_0); and the map μ is essentially the hyperkähler moment map.

Proof of (ii). The equations (1) were studied by Donaldson in [3], and our assertion (ii), as we now explain, can be deduced from an existence result proved there.

We shall consider the complex structure I : the others are not essentially different. Following [3], let us write

$$\begin{aligned}\alpha &= \frac{1}{2}(A_0 + iA_1) \\ \beta &= \frac{1}{2}(A_2 + iA_3);\end{aligned}$$

so α, β take values in \mathfrak{g}^c . The complex structure I now corresponds to multiplication by i in \mathfrak{g}^c . In terms of α, β , the equations (1) can be written

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0 \tag{3a}$$

$$\frac{d}{ds}(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0. \tag{3b}$$

The action of \mathcal{G}_0 on Ω^4 extends to an action of the “complex gauge group”

$$\mathcal{G}_0^c = \{ g : [0, 1] \rightarrow G^c \mid g \in C^2, \ g(0) = g(1) = 1 \}.$$

In terms of (α, β) , the action is

$$\begin{aligned}g(\alpha) &= g\alpha g^{-1} + \frac{1}{2}g \frac{dg^{-1}}{ds} \\ g(\beta) &= g\beta g^{-1}.\end{aligned}$$

This action preserves the “complex equation” (3a), but not the “real equation” (3b). Proposition 2.8 from [3] can be phrased as follows. (In [3] the statement is only made for $G = U(n)$.)

Lemma 2. *If (α, β) satisfy the complex equation (3a), then there is a $g \in \mathcal{G}_0^c$ such that $g(\alpha, \beta)$ satisfies the real equation (3b) also. If g_1 and g_2 both have this property, then $g_1 = fg_2$ for some $f \in \mathcal{G}_0$.*

As a consequence, M can be regarded as the space of solutions of the complex equation modulo the action of \mathcal{G}_0^c . The result is useful because the equation (3a) is trivially integrable: the general solution can be written uniquely in the form

$$\begin{aligned}\alpha &= \frac{1}{2}u \frac{du^{-1}}{ds} \\ \beta &= u\eta u^{-1}\end{aligned} \tag{4}$$

for some path $u : [0, 1] \rightarrow G^c$ with $u(0) = 1$ and some $\eta \in \mathfrak{g}^c$. The action of $g \in \mathcal{G}_0^c$ is to replace u by gu ; so the orbit of (α, β) is uniquely determined by knowledge of the endpoint $u(1) \in G^c$ and of the Lie-algebra element η . This identifies M with $G^c \times \mathfrak{g}^c$ in a holomorphic manner.

If we identify $G^c \times \mathfrak{g}^c$ with T^*G^c much as we did in the opening paragraph (but now using the complex symmetric form $\langle \cdot, \cdot \rangle_c$ on \mathfrak{g}^c) then we obtain an isomorphism

$$M = T^*G^c.$$

The natural holomorphic structure on T^*G^c goes over to the form ω_c on M which is given by

$$\omega_c((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \langle \alpha_1, \beta_2 \rangle_c - \langle \alpha_2, \beta_1 \rangle_c.$$

This completes the proof of (ii).

In Proposition 1 there remains the assertion that $G \times G$ acts on M . Recall that \mathcal{G}_0 lies in the larger group \mathcal{G} , which acts on N also. There is therefore an action of $\mathcal{G}/\mathcal{G}_0$ on $M = N/\mathcal{G}_0$. Evaluation at the endpoints of $[0, 1]$ realizes an isomorphism $\mathcal{G}/\mathcal{G}_0 \rightarrow G \times G$, and the formulae (4) show that the corresponding action of $G \times G$ on T^*G^c is by left- and right-translation.

Remarks. (A) The Riemannian metric on M is complete, because a sequence of smooth solutions to (1) with bounded L^2 -norm has a convergent subsequence after applying gauge transformations. This is most easily seen by using the larger gauge group \mathcal{G} , since each \mathcal{G} -orbit has a representative with $A_0 = 0$, and the equations (1) then bound all the derivatives of A_i in terms of the L^2 -norm. A C^k -convergent subsequence then exists by the Arzela-Ascoli theorem.

(B) As well as the action of $G \times G$, there is an action of $SO(3)$ on M which mixes the three components A_1, A_2, A_3 . This action preserves the metric but mixes the complex structures.

3. The twistor space

The complex structures I, J, K on a hyperkähler manifold M are three members of a family of complex structures parametrized by the imaginary quaternions of unit length:

$$I_x = x_1 I + x_2 J + x_3 K$$

with $\sum x_i^2 = 1$. The twistor space of M (see [8]) is a complex manifold Z with a holomorphic map $\pi : Z \rightarrow S^2 = \mathbb{CP}^1$, such that the fibre $\pi^{-1}(x)$ is the complex manifold (M, I_x) .

We shall examine the twistor space of the manifold M described in the previous section. We showed there that (M, I) is biholomorphic to the cotangent bundle T^*G^c , and the same is true for each complex structure I_x . The twistor space Z will therefore be a holomorphic fibre bundle over \mathbb{CP}^1 with fibre T^*G^c . Let \mathbb{CP}^1 be covered by affine patches U, V with coordinates ζ and $\zeta' = \zeta^{-1}$ respectively. We shall construct trivializations of Z over these patches, giving isomorphisms

$$\begin{aligned}\phi_U : \pi^{-1}(U) &\rightarrow G^c \times \mathfrak{g}^c \times U \\ \phi_V : \pi^{-1}(V) &\rightarrow G^c \times \mathfrak{g}^c \times V.\end{aligned}$$

The twistor space is then completely determined by the transition function $\phi_V \phi_U^{-1}$. We shall show

Proposition 3. *The twistor space of M is the locally trivial fibre bundle Z over \mathbb{CP}^1 expressed as above, with fibre $G^c \times \mathfrak{g}^c$ and transition function*

$$\phi_V \phi_U^{-1} : (g, \eta, \zeta) \mapsto (g \cdot \exp(2\eta/\zeta), \eta\zeta^{-2}, \zeta').$$

Remarks. (A) If T is the total space of the vector bundle $\mathfrak{g}^c \otimes H^2$ over \mathbb{CP}^1 (with H being the Hopf line bundle and H^2 being its tensor-square), the formula above shows Z to be the principal G^c -bundle over T with transition function $\exp(2\eta/\zeta)$. In the case $G = S^1$, this bundle is familiar for the role it plays in the twistor description of magnetic monopoles [4]. That it is the twistor space of $S^1 \times \mathbb{R}^3$ is also well known.

(B) To calculate the hyperkähler metric from the twistor description, one needs some additional data, the most substantial part of which is a family of sections of the projection π . Proposition 3 described the twistor space quite simply, but not the family of sections.

Proof of Proposition 3. The space M was constructed as the hyperkähler quotient of Ω^4 by \mathcal{G}_0 . The quotient construction, quite generally, has a simple interpretation in the twistor picture – something which is explained in [8]. We shall describe how this applies to our case.

Let \mathcal{Z} denote the twistor space of Ω^4 . Since \mathcal{G}_0 acts on Ω^4 , it will act also on \mathcal{Z} , and this action extends to a holomorphic action of the complex group \mathcal{G}_0^c , preserving the fibration over \mathbb{CP}^1 . Each fibre of $\pi : \mathcal{Z} \rightarrow \mathbb{CP}^1$ is a holomorphic-symplectic manifold, the various symplectic forms fitting

together to give a fibre-wise symplectic form taking values in H^2 (see [8]). Accordingly, there is a moment map for the action of \mathcal{G}_0^c on the fibres: in our case it is a map

$$\hat{\mu} : \mathcal{Z} \rightarrow \Gamma^c \otimes H^2. \quad (5)$$

Here Γ^c is the space of C^0 paths in \mathfrak{g}^c , which should be thought of as the continuous part of the dual space of the Lie algebra of \mathcal{G}_0^c . The general principle we now appeal to is that the twistor space of the hyperkähler quotient M is given by

$$Z = \hat{\mu}^{-1}(0)/\mathcal{G}_0^c. \quad (6)$$

Generally we cannot expect this to be true on more than some open set (the set of stable orbits). That it is true globally in our case, even though the spaces concerned are infinite-dimensional, is essentially a restatement of Lemma 2, now applied to the whole family of complex structures I_x . Using (6), we shall prove Proposition 3.

The twistor space of a linear space such as Ω^4 is described in [8], from which we learn that \mathcal{Z} is the vector bundle $\Omega \otimes (H \oplus H)$ over \mathbb{CP}^1 . Let us trivialize \mathcal{Z} over U and V so that

$$\begin{aligned} \mathcal{Z}|_U &= \Omega^c \times \Omega^c \times U \\ \mathcal{Z}|_V &= \Omega^c \times \Omega^c \times V, \end{aligned}$$

where Ω^c is the space of C^1 paths in \mathfrak{g}^c . These trivializations may be chosen so that the transition function identifies $(\alpha, \beta, \zeta) \in \mathcal{Z}|_U$ with $(\alpha', \beta', \zeta') \in \mathcal{Z}|_V$ when

$$\begin{aligned} \alpha' &= \alpha/\zeta \\ \beta' &= \beta/\zeta \\ \zeta' &= 1/\zeta. \end{aligned}$$

The action of \mathcal{G}_0^c on \mathcal{Z} can be written

$$\begin{cases} \alpha \mapsto g\alpha g^{-1} + \frac{1}{2}g \frac{dg^{-1}}{ds} \\ \beta \mapsto g\beta g^{-1} + \frac{1}{2}\zeta g \frac{dg^{-1}}{ds}. \end{cases}$$

In the primed coordinates, this becomes

$$\begin{cases} \alpha' \mapsto g\alpha' g^{-1} + \frac{1}{2}\zeta' g \frac{dg^{-1}}{ds} \\ \beta' \mapsto g\beta' g^{-1} + \frac{1}{2}g \frac{dg^{-1}}{ds}. \end{cases}$$

Over U and V respectively, the moment map $\hat{\mu}$ takes the form

$$\begin{aligned}\hat{\mu} &= \frac{d\beta}{ds} + \zeta \frac{d\alpha}{ds} + 2[\alpha, \beta] \\ \hat{\mu}' &= \frac{d\alpha'}{ds} + \zeta' \frac{d\beta'}{ds} + 2[\alpha', \beta'].\end{aligned}$$

We have $\hat{\mu}' = \hat{\mu}/\zeta^2$ because of the factor H^2 in (5).

For $\zeta \in U$, the general solution to $\hat{\mu} = 0$ can be uniquely written

$$\begin{cases} \alpha = \frac{1}{2}g \frac{dg^{-1}}{ds} \\ \beta = g\eta g^{-1} - \frac{1}{2}\zeta g \frac{dg^{-1}}{ds} \end{cases} \quad (7)$$

for some path $g : [0, 1] \rightarrow G^c$ with $g(0) = 1$, and some $\eta \in \mathfrak{g}^c$. As in §2, the \mathcal{G}_0^c -orbit of this solution is determined by the pair $(g(1), \eta) \in G^c \times \mathfrak{g}^c$. This gives a trivialization

$$\phi_U : Z|_U \rightarrow G^c \times \mathfrak{g}^c \times U.$$

Similarly, for $\zeta' \in V$, the general solution to $\hat{\mu}' = 0$ is

$$\begin{cases} \alpha' = g'\eta'(g')^{-1} + \frac{1}{2}\zeta' g' \frac{d(g')^{-1}}{ds} \\ \beta' = -\frac{1}{2}g' \frac{d(g')^{-1}}{ds} \end{cases} \quad (8)$$

and the data $(g'(1), \eta')$ gives a trivialization

$$\phi_V : Z|_V \rightarrow G^c \times \mathfrak{g}^c \times V.$$

To determine the transition function, suppose (α, β) is written in the form (7), and let $(\alpha', \beta') = (\alpha/\zeta, \beta/\zeta)$. Seeking to express (α', β') in the form (8), we can verify that the solution is to put

$$\begin{aligned}g'(s) &= g(s) \cdot \exp(2s\eta/\zeta) \\ \eta' &= \eta\zeta^{-2}.\end{aligned}$$

At the endpoint $s = 1$, we have the formula for $\phi_V \phi_U^{-1}$ given in Proposition 3. This completes the proof. \square

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Addendum, September 2004

This paper was completed in 1988, while I was at MSRI. The paper was not submitted for publication at the time, and has therefore not been readily available. I have retyped the original manuscript in \TeX , and the present version is the result. No other changes have been made to the original paper.

Perhaps one comment should be added to the original. In the first paragraph of the introduction, it is stated that the complex structure on T^*G which one obtains by identifying it with G^c is compatible with the symplectic structure of this cotangent bundle, in that together they provide a Kähler structure on G^c . No justification for this assertion is given in the text, but it can be easily deduced in the context of the paper by considering G^c as contained in the hyperkähler manifold $M = T^*G^c$ as the locus given by $A_2 = A_3 = 0$.

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